Moduli spaces of points in flags of affine spaces and polymatroids

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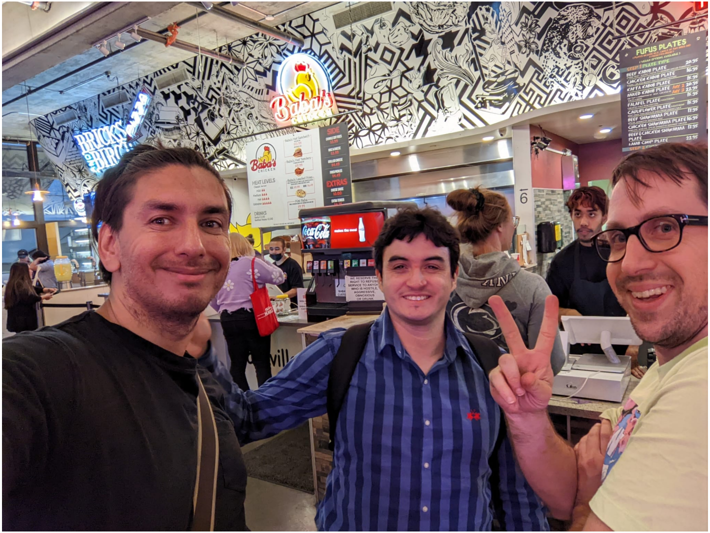
The team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The Fulton-MacPherson compactification

Let X be a smooth variety over $\mathbb{C}$. In 1994 Fulton and MacPherson introduced their compactification of the configuration space

$$\mathcal{U}_n = X^n \setminus \bigcup \Delta_{a,b},$$

where $\Delta_{a,b}$ is the big diagonal where the points labeled by a and b coincide.

Definition and properties

The **Fulton-MacPherson compactification** of $\mathcal{U}_n$, denoted as $X[n]$, is a smooth, normal crossings compactification of $\mathcal{U}_n$. It is endowed with a flat family $\mathcal{F}_n \to X[n]$ whose fibers are degenerations of X marked with n distinct labeled points.

In 2014, Routis generalized this construction by assigning weights $\mathbf{w} = (w_1, \ldots, w_n)$, $0 < w_i \leq 1$, to the points. Routis' compactification parametrizes degenerations of X where points can coincide if their total weight is ≤ 1.

Construction and examples

Construction

Both of these compactifications can be constructed as iterated blow-ups of the "heavy diagonals" in order of non-decreasing dimension.

Examples

- When $X = \mathbb{CP}^1$, the FM compactification is isomorphic to $\overline{M}_{0,n}$.
- If we further points weighted by $\mathbf{w}$, then Routis' compactification is $\overline{M}_{0,\mathbf{w}}$.

A related construction

Consider the deepest diagonal $\Delta \subseteq X^n$ and the following diagram

$$\begin{tikzcd} & D \arrow[r, hook] \arrow[d] & X[n] \arrow[d, "\pi"] \\ x \arrow[r, hook] & \Delta \arrow[r, hook] & X^n, \end{tikzcd}$$

where π is the iterated blow-up construction of $X[n]$.

In 2009 Chen, Gibney and Krashen proved that $\pi^{-1}(x)$ is independent of $x \in \Delta$ and X. Define $T_{d,n} := \pi^{-1}(x)$, where d is the dimension of X.

Theorem (Chen, Gibney, Krashen)

$T_{d,n}$ compactifies the space of configurations of n distinct points on $\mathbb{C}^d$ modulo translation and scaling.

Note: Recall that $M_{0,n}$ parametrizes configurations of n distinct labeled points in $\mathbb{CP}^1$ up to the action of PGL_{n+1}. By considering point configurations such that the $(n+1)th$ point is ∞, we are left with configurations of n points in $\mathbb{C}$ up to translation and scaling. In fact, CGK show that

$$T_{1,n} \cong \overline{M}_{0,n+1}.$$

Weighted version

In 2016 Gallardo and Routis studied the analog construction using the weighted Fulton-MacPherson compactification.

Theorem (Gallardo, Routis)

For any $\mathbf{w} = (w_1, \ldots, w_n)$ with $0 < w_i \leq 1$ such that $w_1 + \cdots + w_n \geq 1$, there is a compactification $T^{\mathbf{w}}_{d,n}$ of the moduli space of n distinct points in $\mathbb{C}^d$ up to translation and scaling.

Examples:

- If $\mathbf{w} = (1, \ldots, 1)$, then $T^{\mathbf{w}}_{d,n}$ this recovers the Chen-Gibney-Krashen compactification.
- If $\mathbf{w} = (1/n, \ldots, 1/n)$, then $T^{\mathbf{w}}_{d,n} = \mathbb{P}^{d(n-1)-1}$.
- If $\mathbf{w} = (\varepsilon, \ldots, \varepsilon, 1)$ for $0 < \varepsilon \ll 1$, then $T^{\mathbf{w}}_{d,n}$ is a toric variety, which we denote $T^{LM}_{d,n}$ and call the **higher-dimensional Losev-Manin compactification**. Moreover, $T^{LM}_{1,n} \cong \overline{M}^{LM}_{0,n+1}$.

The fan of $T^{LM}_{d,n}$

Example

The rays of the fan of $T^{LM}_{2,3}$ are $\{e_1, e_2, e_3, e_4, (1,1,0,0), (0,0,1,1)\}$ in $\mathbb{Z}^4$.

This corresponds to the blow-up of $\mathbb{P}^3$ along two disjoint invariant lines.

Let $N = \mathbb{Z}^{d(n-1)}/(\sum_{i=1}^{n-1}\sum_{k=1}^{d} e_i^k = 0)$ be the lattice of one-parameter subgroups of $\mathbb{P}^{d(n-1)-1}$ with basis $\{\overline{e}_i^k\}$ with $1 \leq i \leq n-1$ and $1 \leq k \leq d$.

Then, the fan of $T^{LM}_{d,n}$ in $N_{\mathbb{R}}$ is generated by the rays

$$\left\{ \overline{e}_i^k \;\middle|\; 1 \leq i \leq n-1, 1 \leq k \leq d \right\}$$

$$\bigcup$$

$$\left\{ \sum_{i \in I} \left(\overline{e}_i^1 + \ldots + \overline{e}_i^d \right) \;\middle|\; 1 \leq |I| \leq n-2,\ I \subsetneq \{1, \ldots, n-1\} \right\}.$$

Structural results

Theorem (Gallardo, —, González, Routis, 2023)

The compactification $T_{d,n}^{LM}$ of the moduli space of n points in $\mathbb{A}^d$ up to translation and scaling satisfies the following:

- $T_{d,n}^{LM}$ is isomorphic to the normalization of a Chow quotient $(\mathbb{P}^d)^{n-1} /\!\!/_{Ch} \mathbb{C}^*$.
- There is a $\delta \subset T_{d,n}^{LM}$ such that the blow-down map $T_{d,n} \to T_{d,n}^{LM}$ factors as
$$T_{d,n} \longrightarrow \mathrm{Bl}_\delta\, T_{d,n}^{LM} \longrightarrow T_{d,n}^{LM},$$
and $\mathrm{Bl}_\delta\, T_{d,n}^{LM}$ is not a Mori dream space for $n \geq 9$.
- $T_{d,n}^{LM}$ is a locally trivial toric fibration over $(\mathbb{P}^{d-1})^{n-1}$, with fiber isomorphic to the Losev-Manin space $\overline{\mathcal{M}}_{0,n+1}^{LM}$.

Higher-dimensional Losev–Manin spaces

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Abstract

The classical Losev–Manin space can be interpreted as a toric compactification of the moduli space of n points in the affine line modulo translation and scaling. Motivated by this, we study its higher-dimensional toric counterparts, which compactify the moduli space of n distinct labeled points in affine space modulo translation and scaling. We show that these moduli spaces are a fibration over a product of projective spaces, that they are isomorphic to the normalization of a Chow quotient, and prove that a related generalization of the moduli space of pointed rational curves proposed by Chen, Gibney, and Krashen is not a Mori dream space for $n \gg 1$. This is joint work with J.L. Gonzalez, P. Gallardo and E. Routis.

The moduli of points in $\mathbb{P}^1$

The moduli space of labelled points in $\mathbb{P}^1$ up to automorphisms is the quasi-projective variety

$$M_{0,n} \cong (\mathbb{P}^1)^{n-3} \setminus \Delta,$$

where Δ is the big diagonal.

This space is not projective, since it doesn't account for collisions between the points. This makes it unsuitable for many applications, for example, it doesn't have a well-defined intersection theory.

Addressing this, Knudsen [13] constructed the moduli space of n-**pointed stable curve of genus** 0 (n-PSC for short), which provides a modular compactification to $M_{0,n}$, denoted as $\overline{M}_{0,n}$. Indeed, $M_{0,n} \subseteq \overline{M}_{0,n}$ since an n-pointed $\mathbb{P}^1$ is an n-PSC.

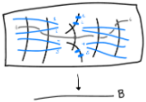

Figure 1. A family with an n-pointed stable curve of genus 0 as central fiber.

Two decades later, Hassett generalized this construction by allowing collisions:

Weight data

To each of the n points assign the weight weight datum $w_i \in (0,1] \cap \mathbb{Q}$, where $\sum w_i > 2$. For any fixed weight datum, one defines an n-**pointed weighted stable curve** to be an n-PSC such that a collection of points can collide if and only if the sum their of weights is < 1.

Theorem [10]

Consider the weight datum $\mathcal{A}$. The moduli space of the n-pointed weighted stable curves, denoted by $\overline{M}_{0,\mathcal{A}}$, is a fine moduli space compactifying $M_{0,n}$.

Examples

- If $\mathcal{A} = (1,\dots,1)$, then one recovers $\overline{M}_n \cong \overline{M}_{0,n}$.
- If $\mathcal{A} = (1,1,\varepsilon,\dots,\varepsilon)$ for $\varepsilon \ll 1$, then $\overline{M}_{0,\mathcal{A}}$ is a toric variety called the **Losev–Manin space**, denoted $\overline{M}_{0,n}^{LM}$. Its normal fan is the normal fan of the permutohedron [14].

Theorem [3]

Every toric Hassett space $\overline{M}_{0,\mathcal{A}}$ is a toric blow-down of $\overline{M}_{0,n}^{LM}$.

The moduli space of points in affine space

The moduli space of points in $\mathbb{A}^d$ up to translation and scaling is not compact. L. Chen, A. Gibney and D. Krashen compactified it by considering n-**pointed stable rooted trees** [2].

An n-**pointed stable rooted tree** (n-PSRT) is a variety X such that

1. The dual intersection complex V of X is a rooted tree.
2. If $v \in V$ is a leaf, then the irreducible component $X_v \subseteq X$ is isomorphic to $\mathbb{P}^d$. Otherwise, it is isomorphic to the blow-up of $\mathbb{P}^d$ at points.
3. Each $X_v \subseteq X$ is equipped with a fixed hyperplane $H_v \subseteq X_v$ disjoint from any exceptional divisors.
4. The intersection of two adjacent components is a hyperplane in one of them and the exceptional divisor in the other.
5. Each X_v must have at least two different markings, all disjoint from H_v and the exceptional divisors; here a marking is either a marked point or an irreducible exceptional divisor. The hyperplane $H_v \subseteq X_v$ is not considered a marking of X_v.

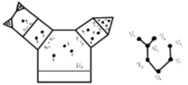

Figure 2. An n-pointed stable rooted tree X and its tree graph V.

$T_{d,n}$ as a generalization of $\overline{M}_{0,n}$ [2]

There is an isomorphism $T_{d,n} \cong \overline{M}_{0,n}$. Indeed, a 1-dimensional n-PSRT is exactly an $(n+1)$-pointed stable curve, where the root hyperplane is identified with the $(n+1)$st marking.

P. Gallardo and E. Routis generalized the construction by endowing the points with weight data $\mathcal{A} = (1, a_2, \dots, a_n)$ [8]. Their moduli spaces parametrize weighted n-PSRT, which in addition satisfy that:

6. Any number of marked points in an irreducible component $X_v \subseteq X$ are allowed to collide provided that the sum of their weights remains ≤ 1.
7. The point of weight 1 lies in a leaf component.

Examples

- If $\mathcal{A} = (1,\dots,1)$, then $T_{d,n}^{\mathcal{A}} \cong T_{d,n}$.
- If $\mathcal{A} = (1,\varepsilon,\dots,\varepsilon)$, for $\varepsilon \ll 1$, then $T_{d,n,\mathcal{A}}$ is a toric variety called the **higher-dimensional Losev–Manin space**, denoted $T_{d,n}^{LM}$.

Proposition

$T_{d,n}^{LM}$ is the toric variety associated to the fan $\Delta_{T_{d,n}^{LM}}$ in

$$N \otimes \mathbb{R} = \mathbb{R}^{d(n-1)} / \Big(\sum_{i=1}^{n-1} \sum_{k=1}^{d} e_i^k = 0\Big)$$

whose rays are generated by the vectors

$$\left\{ \overline{e}_i^k \;\middle|\; i \in [n-1], k \in [d] \right\} \cup \left\{ \sum_{i \in I} \left(\overline{e}_i^1 + \ldots + \overline{e}_i^d \right) \;\middle|\; 1 \leq |I| \leq n-2,\ I \subsetneq [n-1] \right\}.$$

Theorem [7, Gallardo, —, Gonzalez, Routis]

The smooth toric compactification $T_{d,n}^{LM}$ satisfies the following:

1. $T_{d,n}^{LM}$ is isomorphic to the normalization of a Chow quotient $(\mathbb{P}^d)^{n-1} /\!\!/_{Ch} \mathbb{C}^*$.
2. There is an irreducible closed locus $\delta \subset T_{d,n}^{LM}$ with $\dim(\delta) = d-1$ such that the canonical map $T_{d,n} \to T_{d,n}^{LM}$ factors as
$$T_{d,n} \longrightarrow \mathrm{Bl}_\delta T_{d,n}^{LM} \longrightarrow T_{d,n}^{LM},$$
and $\mathrm{Bl}_\delta T_{d,n}^{LM}$ is not a Mori dream space (MDS) for $n \geq 9$.
3. $T_{d,n}^{LM}$ is a locally trivial fibration over $(\mathbb{P}^{d-1})^{n-1}$, with fiber isomorphic to the Losev–Manin space $\overline{M}_{0,n+1}^{LM}$.

Remarks

- Part 1 enhances the functorial interpretation of $T_{d,n}^{LM}$ as a moduli space. Our Chow quotient is a closure in a Chow variety, which represents a functor of certain families of cycles. This represents an improvement over the initial constructions in [2, 8], where $T_{d,n}$ and its weighted variants were obtained from the Fulton–MacPherson compactification $\mathbb{P}^d[n]$ [5].
- Much attention has been given to the birational geometry of blow-ups of toric varieties at a point; see for example. This is crucial in proving that $\overline{M}_{0,n}$ is not a Mori dream space for $n \gg 1$, [1, 9, 11]. However, research on blow-ups with higher-dimensional centers is limited. To address this, use the work of [12] to establish the following result.

Theorem [7, Gallardo, —, Gonzalez, Routis]

Let X be a complete toric variety and let T be a subtorus of its torus. Let Y be a complete toric variety such that the rays of its fan are obtained by projecting the rays of the fan of X modulo the lattice of one-parameter subgroups of T. Let Z be the closure of T on X and e be a point in the torus of Y. Then,

$$\mathrm{Cox}(\mathrm{Bl}_Z X) \text{ is finitely generated} \iff \mathrm{Cox}(\mathrm{Bl}_e Y) \text{ is finitely generated.}$$

The blow-up $\mathrm{Bl}_\delta T_{d,n}^{LM}$ fits precisely the hypothesis of this theorem for $Z = \delta$ and $X = T_{d,n}^{LM}$. In this case, the space Y is a related moduli space, whose blow-up at e is known not the be a MDS as soon as $n \geq 9$, independently of n. See [6]

Corollary

The space $T_{d,n}$ is not a Mori dream space for $n \geq 9$.

Remarks

The space $T_{d,2}$ is a MDS because it is isomorphic to a projective space, $T_{1,n}$ is a MDS for $n \leq 5$ because it is isomorphic to $\overline{M}_{0,n+1}$, and $T_{2,3}$ is the blow-up of three disjoint lines in $\mathbb{P}^3$, so it is a MDS by [4, Theorem 4.4].

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Generalizing this story

We construct three different moduli spaces related to configurations of distinct ordered points in a flag of affine spaces

$$\mathbb{C}^{a_1} \supseteq \mathbb{C}^{a_2} \cdots \supseteq \mathbb{C}^{a_n}.$$

- A weighted Fulton-MacPherson-type compactification.
- A weighted compactification for configurations defined up to translation and scaling, generalizing Chen-Gibney-Krashen's moduli spaces of points in affine space up to translation and scaling.
- A compactification where points are allowed to collide, related to Kim-Sato's generalized Fulton-MacPherson.

Toric compactifications

These moduli spaces admit toric compactifications generalizing the Losev-Manin space. Their corresponding polytopes are polymatroids.

- Introduced by Edmonds in the 70s in connection with combinatorial optimization.
- Generalize matroids.
- Model subspace arrangements instead of only hyperplane arrangements.

Setting

Definition

Let $[n] := \{1, \ldots, n\}$, and consider a surjective map $\pi : A \to [n]$ of finite sets. The n-tuple

$$\mathbf{a} := (a_1, \ldots, a_n), \quad a_i = |\pi^{-1}(i)|$$

is the **cage** or **caging** associated to π.

- Suppose $a_1 \geq \cdots \geq a_n$. A cage defines a flag of vector spaces:

$$\mathbb{C}^{a_1} \supseteq \mathbb{C}^{a_2} \supseteq \cdots \supseteq \mathbb{C}^{a_n}.$$

- A collection of points in the flag is an element of $\mathbb{C}^{\mathbf{a}} := \mathbb{C}^{a_1} \times \mathbb{C}^{a_2} \times \cdots \times \mathbb{C}^{a_n}$.

Parametrization problem: Collections of n ordered *distinct* points $(p_1, \ldots, p_n) \in \mathbb{C}^{\mathbf{a}}$.

These are parametrized by

$$\mathbb{C}^{\mathbf{a}} \setminus \bigcup_{I \subseteq [n], |I| \geq 2} \Delta_I^{\mathbf{a}},$$

where

$$\Delta_I^{\mathbf{a}} = \{p \in \mathbb{C}^{\mathbf{a}} \mid p_i = p_j \, \forall i, j \in I\}$$

Theorem 1 (Gallardo, — , González)

There exists a smooth, normal crossings, *geometric* compactification $T^{\mathbf{a}}_{\mathbf{w}}$ of the moduli space of n distinct labeled points in the flag $\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$, up to scaling and translation along $\mathbb{C}^{a_n}$.

(1) If $\mathbf{w} = (\varepsilon, \ldots, \varepsilon, 1)$ with $\varepsilon \ll 1$, then $T^{\mathbf{a}}_{LM} := T^{\mathbf{a}}_{\mathbf{w}}$ is isomorphic to the polypermutohedral toric variety associated to $\mathbf{a}$. We refer to this as the *Losev-Manin compactification*.

(2) Let $T^{\mathbf{a}}$ denote the compactification corresponding to the weight $\mathbf{w} = (1, \ldots, 1)$. Just as in Kapranov's construction of $\overline{M}_{0,n}$, there is a sequence of smooth blow-ups

$$T^{\mathbf{a}} \longrightarrow T^{\mathbf{a}}_{LM} \longrightarrow \mathbb{P}^{a_1 + \ldots + a_{(n-1)} - 1}.$$

(3) $T^{\mathbf{a}}_{LM}$ is a non-trivial, locally trivial fibration

$$\begin{tikzcd} \overline{M}^{LM}_{0,n+1} \arrow[r] & T^{\mathbf{a}}_{LM} \arrow[d] \\ & \prod_{i=1}^{n-1} \mathbb{P}^{a_i - 1} \end{tikzcd}$$

Remark: If $\mathbf{a} = (d, \ldots, d)$, then $T^{\mathbf{a}}_{\mathbf{w}} = T^{\mathbf{w}}_{d,n}$, Gallardo-Routis' moduli spaces. If $d = 1$, then $T^{\mathbf{a}}_{\mathbf{w}} = \overline{M}_{0,\mathbf{w} \cup \{1\}}$.

Theorem 2 (Gallardo, —, González)

Consider an n-tuple $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$. There exist a smooth, normal crossings, *geometric* compactification $\mathbb{P}^{[\mathbf{a}]}_H$ of the configuration space of n non-necessarily distinct points in a flag of affine spaces such that:

(1) The variety $\mathbb{P}^{[\mathbf{a}]}_H$ is constructed as an iterated blow-up of $\prod_{i=1}^n \mathbb{P}^{a_i}$ along torus invariant subvarieties. In particular, it is a toric variety itself.

(2) $\mathbb{P}^{[\mathbf{a}]}_H$ is isomorphic to the polystellahedral variety associated to $\mathbf{a}$.

(3) There exist an open $\left(\mathbb{P}^{[\mathbf{a}]}_H\right)^\circ \subsetneq \mathbb{P}^{[\mathbf{a}]}_H$ and a geometric quotient such that

$$T^{\mathbf{a}}_{LM} \cong \left(\mathbb{P}^{[\mathbf{a}]}_H\right)^\circ //\mathbb{G}_m.$$

Proof sketches

- The main ingredient in the construction of these moduli spaces is Li Li's theory of wonderful compactifications. This gives an explicit construction of our spaces as iterated blow-ups along a *building set*.
- The universal families are constructed by embedding our spaces in other known compactifications—Routis' weighted Fulton-MacPherson and Kim-Sato's generalized Fulton-MacPherson.
- Toric compactifications are obtained by identifying the torus-invariant strata in the building set.
- Identifying the spaces with the polypermutohedral and polystellahedral varieties requires some hard work.

Constructing the toric compactifications

The polypermutohedral and polystellahedral varieties correspond to polytopes known as polypermutohedron and polystellahedron. These are examples of nestohedra (easy to work with in general!)

Definition

A combinatorial building set over the set $[n]$ is a collection $\mathcal{B} \subseteq 2^{[n]} \setminus \emptyset$ such that

- $\{i\} \in \mathcal{B}$ for all $i \in [n]$;
- If $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$.

Nestohedra are polytopes constructed from combinatorial building sets.

Start with the inner normal fan Σ_{n-1} of the $(n-1)$-simplex, whose facets have been labeled by $[n]$. Then, take the star subdivision along the cones $\sigma_I \in \Sigma_{n-1}$ for $I \in \mathcal{B}$ in such a way that larger sets come first.

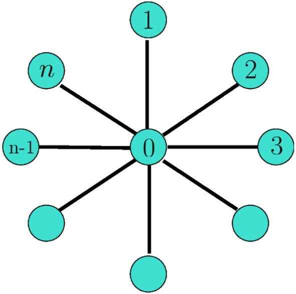
Examples

Example 1: The permutohedron is the nestohedron of the building set $\mathcal{B}_{LM} = 2^{[n]} \setminus \emptyset$. Its corresponding toric variety is isomorphic to Losev-Manin's space $\overline{M}_{0,n+2}$.

Example 2: Let G be the star graph with vertices $\{0, \ldots, n\}$. Define the building set

$$\mathcal{B}_{Star} = \{I \subset \{0, \ldots, n\} \mid G|_I \text{ is connected}\}.$$

The stellahedron is the nestohedron of this building set.



Pullback of building sets

Polypermutohedra and polystellahedra are defined as "expansions" of permutohedra and stellahedra by Crowley-Huh-Larson-Simpson-Wang and Eur-Larson. We rephrase this in terms of *pullbacks of a building sets*.

Definition

Let $\mathcal{B}$ be a building set on $[n]$ and consider a function $\pi : A \to [n]$ of finite sets. The pullback of $\mathcal{B}$ along π is the building set on A defined as

$$\pi^*\mathcal{B} = \{\{i\} \mid i \in A\} \cup \{\pi^{-1}(I) \mid I \in \mathcal{B}\}.$$

- The polypermutohedron with cage $\mathbf{a}$ is the nestohedron corresponding to the pullback of $\mathcal{B}_{LM}$ along a caging $\pi : A \to [n]$ with cage $\mathbf{a}$.
- The polystellahedron with cage $\mathbf{a}$ is the nestohedron corresponding to the pullback of $\mathcal{B}_{Star}$ along a caging $\pi : A \cup \{0\} \to [n] \cup \{0\}$ such that $\pi^{-1}(0) = \{0\}$.
- These are the base and independence polytopes, resp., of a polymatroid.

Example

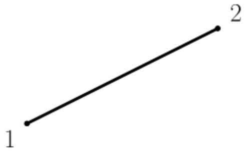
Consider the building set $\mathcal{B}_{LM} = 2^{[4]} \setminus \emptyset$, and the caging

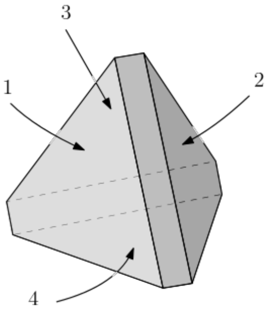
$$\pi : \{1,2,3,4\} \to \{1,2\}, \qquad \pi^{-1}(1) = \{1,2\}, \pi^{-1}(2) = \{3,4\}.$$

Then,

$$\mathcal{B} = \{\{1,2\},\{3,4\},\{1,2,3,4\}\} \cup \text{Singletons}.$$

The resulting polytope is the polypermutohedron with cage $(2,2,2)$.



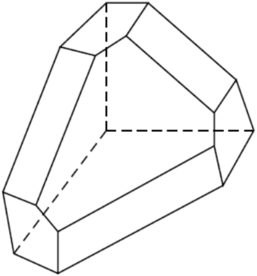


PP as GIT quotient of PS

The isomorphism

$$T_{LM}^{\mathbf{a}} \cong \left(\mathbb{P}_H^{[\mathbf{a}]}\right)^{\circ} //\mathbb{G}_m,$$

where $T_{LM}^{\mathbf{a}}$ and $\mathbb{P}_H^{[\mathbf{a}]}$ are the polypermutohedral and polystellahedral varieties with cage **a** follows from the known fact that the polypermutohedron with cage **a** is a facet of the polystellahedron with cage **a**.



Consider any n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ and $m = \sum a_i$. Let $\mathbf{1}^m$ denote the weight vector with m ones.

It follows from the pullback construction that the polypermutohedron and polystellahedron with cage $\mathbf{1}^m$ are permutohedra and stellahedra.

$$\begin{tikzcd}
\mathbb{P}_H^{[\mathbf{1}^m]} \arrow[r, dotted, "/\!\!/\mathbb{C}^*"] \arrow[d] & \overline{M}_{0,m+2}^{LM} \arrow[d] & \overline{M}_{0,m+2} \arrow[l] \arrow[d, dotted] \\
\mathbb{P}_H^{[\mathbf{a}]} \arrow[r, dotted, "/\!\!/\mathbb{C}^*"] \arrow[d] & T_{LM}^{\mathbf{a}} \arrow[d] & T^{\mathbf{a}} \arrow[l] \\
\prod_{i=1}^n \mathbb{P}^{a_i} & \mathbb{P}^{a_1+\cdots+a_{(n-1)}-1} &
\end{tikzcd}$$